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RULES FOR THE ALGEBRAIC SIGNS OF HYPERBOLIC FORMULÆ.

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In Greenhill's Calculus large use is made of Hyperbolic Functions, in which the sine and cosine of an angle of a rectangular hyperbola (marked $\sinh u$ and $\cosh u$) are expressed in exponential form by $\frac{1}{2}(e^u - e^{-u})$ and $\frac{1}{2}(e^u + e^{-u})$. These values and the formulæ derived from them differ from the usual trigonometrical formulæ only by the absence from their exponential expressions of the imaginary $i (= \sqrt{-1})$, and by changes of algebraic sign consequent on this absence. We may therefore convert all trigonometrical or circular formulæ into hyperbolic formulæ, by developing them in exponential form and cancelling the imaginary whenever it appears. As both trigonometrical and hyperbolic formulæ are only particular cases of the more general elliptic functions, the hyperbolic formulæ do not enable us to accomplish anything that cannot be otherwise effected. But they have the advantage of frequently simplifying operations in calculus.

No convenient rules, however, have, so far as we know, been offered for enabling us to decide by inspection when we should change the algebraic sign in writing out the hyperbolic formulæ after the pattern of the trigonometrical; and the mere effort of memory required for this in the absence of rules is confusing to the mind when we are applying the formulæ in calculations. We therefore submit two rules which will, we think, cover all the cases; premising that the sine-factor of an angle is present in its sine, tangent, cotangent, and cosecant, but not in its cosine or secant.

RULE I. When a term of a trigonometrical formula contains two or three sine-factors, or any number from the series 2, 3, 6, 7, etc., the corresponding term of a hyperbolic formula changes its algebraic sign; and not in other cases.

EXAMPLES—1. From

$$1 = \cos^2 \theta + \sin^2 \theta$$

we may derive by inspection the hyperbolic formula

$$1 = \cosh^2 u - \sinh^2 u.$$

2. From

$$\tan(\theta \pm \varphi) = \frac{\tan \theta \pm \tan \varphi}{1 \mp \tan \theta \tan \varphi},$$

we derive

$$\tanh(u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}.$$

3. As an example with three sine-factors we may take

$$4 \sin^3 \theta = -\sin 3\theta + 3 \sin \theta ;$$

whence we derive the hyperbolic formula

$$-4 \sinh^3 u = -\sinh 3u + 3 \sinh u .$$

The reason of this rule is, that since (assuming $u = i\theta$)

$$i \sin \theta = \frac{1}{2} (e^{i\theta} - e^{-i\theta}) = \sinh u ,$$

the product of two or three sets of sine-factors on each side of this equality (e. g. $i \sin \theta . i \sin \varphi . i \sin \chi = \sinh u . \sinh v . \sinh w$), will render the trigonometrical and the hyperbolic terms of contrary algebraic signs, whilst the product of four or five such factors will keep the signs identical. Nor can any number of cosine factors influence the result, as they do not contain an imaginary in the coefficient of their exponential expression.

RULE II. In direct differentiation, when the differential of a term of a trigonometrical function contains one or two sine-factors, or any number from the series 1, 2, 5, 6, etc., the differential of the corresponding term of a hyperbolic function changes its algebraic sign ; and not in other cases.

Thus, we have

$$\frac{d \sinh u}{du} = \cosh u ,$$

the same algebraic sign as in trigonometry ; but

$$\frac{d \cosh u}{du} = \sinh u$$

has its sign changed. From the trigonometrical differentiation of

$$\frac{d \sec \theta}{d\theta} = \tan \theta \sec \theta ,$$

we derive by inspection under Rule II, the hyperbolic

$$\frac{d \operatorname{sech} u}{du} = -\tanh u \operatorname{sech} u .$$

The reason of this rule is seen by comparing the corresponding developments.

Thus the trigonometrical function

$$\frac{d}{d\theta} \left(\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right) = \frac{d}{d\theta} \left[\frac{1}{2} \theta - \frac{1}{4} \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right] = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i} \right]^2 = \sin^2 \theta ;$$

whilst the hyperbolic function

$$\frac{d}{du} \left(\frac{1}{2}u - \frac{1}{4} \sinh 2u \right) = \frac{d}{du} \left[\frac{1}{2}u - \frac{1}{4} \frac{e^{2u} - e^{-2u}}{2} \right] = - \left[\frac{e^u - e^{-u}}{2} \right]^2 = - \sinh^2 u.$$

Here we observe that the difference between Rules I and II depends on the fact that in differentiating the trigonometrical functions an imaginary factor is imported from the index of the exponentials into the coefficient. When only a cosine-factor comes out in the differential the imported imaginary cancels one previously in the denominator.

According to Rule II,

$$\frac{d}{du} \left(\frac{1}{2} \cosh^3 u - \cosh u \right) = \sinh^3 u,$$

with the same signs as in the trigonometrical function, thus contrasting with example 3 under Rule I.

Rule I generally applies as to the differentiation of *Inverse Functions*, where the variable includes in itself a sine-factor. Thus, from

$$\frac{d}{dx} \sin^{-1} \frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$$

we obtain by inspection, for the hyperbolic function,

$$\frac{d}{dx} \sinh^{-1} \frac{x}{a} = \frac{1}{\sqrt{a^2 + x^2}};$$

and from

$$\frac{d}{dx} \tan^{-1} \frac{x}{a} = \frac{a}{a^2 + x^2},$$

we derive

$$\frac{d}{dx} \tanh^{-1} \frac{x}{a} = \frac{a}{a^2 - x^2}.$$

In these cases the change of sign enters in course of reduction, and not of differentiation. But the differentiation of $\cosh^{-1} x/a$ involves both rules. Thus assuming $a^{-1}x = \cosh u$, we have, by Rule II,

$$\frac{1}{a} \frac{dx}{du} = \sinh u.$$

But, by Rule I,

$$\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{a^{-2}x^2 - 1} = a^{-1} \sqrt{x^2 - a^2}:$$

Hence

$$\frac{d}{dx} \cosh^{-1} \frac{x}{a} = \frac{du}{dx} = \frac{1}{\sqrt{x^2 - a^2}}.$$

Comparing this with the trigonometrical

$$\frac{d}{dx} \cos^{-1} \frac{x}{a} = - \frac{1}{\sqrt{a^2 - x^2}},$$

we see how it can be derived by inspection. Because the whole differential contains a sine-factor in its denominator we change its algebraic sign according to Rule II; and the algebraic sign of the part under the radical, having the square of a sine-factor is changed by Rule I.

Note.—Assuming the hyperbolic angle u to be the ghost or shadow (the anti-Gudermannian) of the circular angle $i\theta$, the rule for passing in values from one to the other order of formulæ is graphically exhibited by the parallelism of the two formulæ

$$1 = \sec^2 \theta - \tan^2 \theta, \quad 1 = \cosh^2 u - \sinh^2 u;$$

in which

$$\sec \theta = \cosh u, \quad \text{and} \quad \tan \theta = \sinh u.$$

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